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# DETERMINATION OF THE GEOMETRIC CHARACTERISTICS OF SHELLS OF REVOLUTION SATISFYING FINITE STRAIN COMPATIBILITY CONDITIONS

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Taking account of shear strain, formulas to determine the Lamé coefficients, and the principal radii of curvature of coordinates of points of the middle surface of an arbitrary shell of revolution after it has been loaded by an axisymmetric load, are obtained in a quadratic approximation. The Lamé coefficients and the radii of curvature of the deformed shell satisfy Codazzi — Gauss conditions and their associated compatibility equations for finite strain.

It is shown that the finite strain compatibility equations obtained in this paper from the condition that the strain incompatibility tensor [1] of a three-dimensional solid equals zero, are satisfied identically if the small strain components satisfy the appropriate linear equations of strain compatibility of the shell middle surface.

**1. Finite strain compatibility conditions of the middle surface of shells of revolution.** We start from the fact that the strain incompatibility tensor, equal to the difference between the Riemann — Christoffel tensors in the states of strain $R_{krmn}$ and no strain $r_{krmn}$ is zero

$$R_{krmn} - r_{krmn} = 0 \tag{1.1}$$

Here and henceforth, the capital Latin letters will denote functions of the state of strain, while the analogous lower case letters will denote the state of no strain. The subcripts $k, r, m, n$ and others will run through values from 1 to 3.

The components of the Riemann — Christoffel tensor are expressed in terms of Christoffel symbols of the first kind $P_{krm}$ and components of the metric tensor $G^{\alpha\beta}$ as follows [2]:

$$R_{krmn} = \frac{\partial P_{nmk}}{\partial x^r} - \frac{\partial P_{nmr}}{\partial x^k} + G^{\alpha\beta}(P_{\beta mr}P_{\alpha nk} - P_{\beta mk}P_{\alpha nr}) \tag{1.2}$$

($x^r$ are material coordinates of points of the medium; summation from 1 to 3 is carried out over the repeated subscripts). The tensor $r_{krmn}$ has an analogous form.

Defining the finite strain tensor $\xi_{kr}$ as half the difference between the metric tensor components [2]

$$\xi_{kr} = {}^1/_2\,(G_{kr} - g_{kr}) \tag{1.3}$$

we rewrite the compatibility equations (1.1) in the form

$$\frac{\partial \xi_{nmk}}{\partial x^r} - \frac{\partial \xi_{nmr}}{\partial x^k} + G^{\alpha\beta}\,[(\xi_{\beta mr} + p_{\beta mr})(\xi_{\alpha nk} + p_{\alpha nk}) - \tag{1.4}$$

$$(\xi_{\beta mk} - p_{\beta mk})(\xi_{\alpha nr} + p_{\alpha nr})] - g^{\alpha\beta}(p_{\beta mr}p_{\alpha nk} - p_{\beta mk}p_{\alpha nr}) = 0$$

$$\xi_{nmk} = P_{nmk} - p_{nmk} = \frac{\partial \xi_{mn}}{\partial x^k} + \frac{\partial \xi_{nk}}{\partial x^m} - \frac{\partial \xi_{mk}}{\partial x^n} \tag{1.5}$$

$$P_{nmk} = \frac{1}{2}\left(\frac{\partial G_{nm}}{\partial x^k} + \frac{\partial G_{nk}}{\partial x^m} - \frac{\partial G_{mk}}{\partial x^n}\right), \quad p_{nmk} = \frac{1}{2}\left(\frac{\partial g_{nm}}{\partial x^k} + \frac{\partial g_{nk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^n}\right)$$

The contravariant components of the metric tensor $G^{\alpha\beta}$ are defined as elements of the inverse matrix $G_{\alpha\beta}$ [2]:

$$\| G^{\alpha\beta} \| = \| G_{\alpha\beta} \|^{-1} = \| g_{\alpha\beta} + 2\xi_{\alpha\beta} \|^{-1}$$

The elements of this matrix can be obtained as follows [1]:

$$G^{\alpha\beta} = \frac{1}{G}\frac{\partial G}{\partial G_{\beta\alpha}}, \quad G = \det \| G_{\alpha\beta} \| \tag{1.6}$$

Substituting (1.6) into (1.4) permits getting rid of the metric tensor components in the state of strain in the compatibility equations.

A further conversion of the compatibility equations (1.4) in general form is inexpedient because of their awkwardness. Let us convert these equations for axisymmetrically deformable shells whose material coordinates are mutually orthogonal in the initial state. In this case

$$g_{12} = g_{23} = g_{13} = 0, \quad G_{12} = G_{23} = 0, \quad \xi_{12} = \xi_{23} = 0$$

Introducing the Lamé coefficients $h_{(k)}$ for orthogonal coordinates [1]

$$g_{11} = h_{(1)}{}^2, \; g_{22} = h_{(2)}{}^2, \; g_{33} = h_{(3)}{}^2 \tag{1.7}$$

we express the strain tensor components in terms of their physical components $\xi_{(kr)}$

$$\xi_{kr} = h_{(k)}h_{(r)}\xi_{(kr)} \tag{1.8}$$

Six Lamé relationships [1] were used in converting the compatibility equations (1.4). Under the assumption of a linear distribution law for the Lamé coefficients over the thickness for axisymmetric shells of revolution, these relationships reduce to two Codazzi — Gauss relations

$$\frac{d}{d\alpha_1}\left(\frac{h_2}{r_2}\right) - \frac{1}{r_1}\frac{dh_2}{d\alpha_1} = 0, \quad \frac{d}{d\alpha_1}\left(\frac{1}{h_1}\frac{dh_2}{d\alpha_1}\right) + \frac{h_1h_2}{r_1r_2} = 0 \tag{1.9}$$

$$\left(h_{(1)} = h_1\left(1 + \frac{z}{r_1}\right), \quad h_{(2)} = h_2\left(1 + \frac{z}{r_2}\right), \; h_{(3)} = 1\right)$$

Here $h_1, h_2$ are Lamé coefficients of the shell middle surface, $r_1, r_2$ are its radii of curvature, $\alpha_1 = x^1$ is the curvilinear coordinate directed along the shell generator, $z = x^3$ is the rectilinear coordinate orthogonal to the middle surface of the undeformed shell (this latter is shown by curve 1 in the Fig. 1 and directed toward its outer normal (Fig. 1).

The second relationship in (1.9) becomes an identity for a shell of revolution, upon compliance with the first.

Let us refer the deformations $\xi_{(kr)}$ to the shell middle surface.

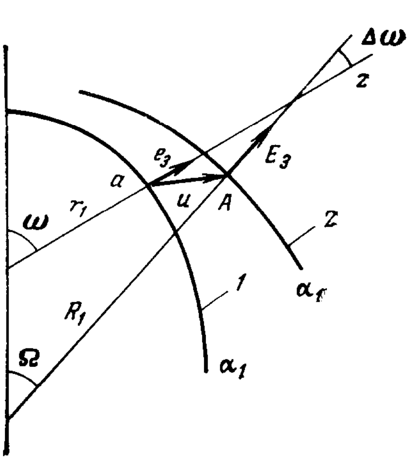


Fig. 1

If a linear distribution of the displacements over the shell thickness is used, the following dependences can be obtained between the finite strains of an arbitrary point of the shell $\xi_{(kr)}$ on one hand, and the finite strains of the middle surface $\varepsilon_{kr}$ and changes in the curvatures $\chi_{kr}$, on the other:

$$\begin{aligned} &\xi_{(11)}\left(1+\frac{z}{r_1}\right)=\varepsilon_{11}+z\chi_{11} \\ &\xi_{(22)}\left(1+\frac{z}{r_2}\right)=\varepsilon_{22}+z\chi_{22} \\ &\xi_{(13)}\left(1+\frac{z}{r_1}\right)=\varepsilon_{13}+z\chi_{13} \\ &\xi_{(33)}=\varepsilon_{33},\ \chi_{33}=0 \end{aligned} \qquad (1.10)$$

The components of the finite strain tensor of the middle surface $\varepsilon_{kr}$ and $\chi_{kr}$ can be expressed in terms of the components of the small strain tensor $e_{kr}$ and the change in the curvatures $\varkappa_{kr}$ if (3.9.1) is used (see [1]. It hence follows ( $\vartheta$ is the angle of normal rotation to the shell middle surface)

$$\begin{aligned} &\varepsilon_{11}=e_{11}+{}^1/_2\,[e_{11}{}^2+(\vartheta-2e_{13})^2], \qquad \varepsilon_{22}=e_{22}+{}^1/_2 e_{22}{}^2 \\ &\varepsilon_{33}={}^1/_2\vartheta^2,\ \varepsilon_{13}=e_{13}+{}^1/_2\ e_{11}\vartheta \\ &\chi_{11}=\varkappa_{11}+\frac{1}{2r_1}\left[e_{11}(2r_1\varkappa_{11}-e_{11})+\vartheta^2-4e_{13}^2\right] \\ &\chi_{22}=\varkappa_{22}+e_{22}\left(\varkappa_{22}-\frac{e_{22}}{2r_2}\right),\ \ \chi_{13}=\frac{1}{2}\varkappa_{11}\vartheta \end{aligned} \qquad (1.11)$$

The remaining components of $\varepsilon_{kr}$ and $\chi_{kr}$ vanish. It can be shown, and it is done so in [3], for instance, that for axisymmetric shells

$$\varkappa_{11}=\frac{1}{h_1}\frac{d\vartheta}{d\alpha_1},\qquad \varkappa_{22}=\frac{\vartheta}{h_1h_2}\frac{dh_2}{d\alpha_1} \qquad (1.12)$$

Substituting the relationships (1.5) and (1.6) into the strain compatibility equations (1.4), referring the expressions obtained after this to the shell middle surface by using (1.7)-(1.10), and limiting ourselves to a quadratic approximation, we arrive at two differential relationships

$$\begin{aligned} &\frac{d}{d\alpha_1}\left[-\frac{1}{h_1}\frac{d}{d\alpha_1}(h_2\varepsilon_{22})+\frac{\varepsilon_{11}}{h_1}\frac{dh_2}{d\alpha_1}+2\frac{h_2}{r_2}\varepsilon_{13}\right]+\frac{h_1h_2}{r_1r_2}(2\varepsilon_{33}-r_1\chi_{11}-r_2\chi_{22})+ \\ &\frac{h_1h_2}{r_1r_2}\left[2\varepsilon_{33}(\varepsilon_{11}+\varepsilon_{22}-2\varepsilon_{33}+r_1\chi_{11}+r_2\chi_{22})-(\varepsilon_{11}+r_1\chi_{11})\times\right. \\ &\left.(\varepsilon_{22}+r_2\chi_{22})-4\varepsilon_{13}^2\right]+\frac{2}{h_1}\frac{dh_2}{d\alpha_1}\left[(\varepsilon_{22}-\varepsilon_{11})\frac{d\varepsilon_{11}}{d\alpha_1}+\frac{h_1}{r_1}\varepsilon_{13}(r_1\chi_{11}-\right. \\ &\left.\varepsilon_{11}+2\varepsilon_{22}-2\varepsilon_{33})-2\varepsilon_{13}\frac{d\varepsilon_{13}}{d\alpha_1}\right]+\frac{h_2}{h_1}\frac{d\varepsilon_{22}}{d\alpha_1}\frac{d}{d\alpha_1}(\varepsilon_{11}+\varepsilon_{22})+ \\ &2h_2\varepsilon_{13}\left(\frac{1}{r_1}\frac{d\varepsilon_{22}}{d\alpha_1}+\frac{1}{r_2}\frac{d\varepsilon_{11}}{d\alpha_1}\right)+2\frac{h_2}{r_2}\frac{d\varepsilon_{13}}{d\alpha_1}(r_2\chi_{22}-\varepsilon_{22}-2\varepsilon_{33})=0 \end{aligned} \qquad (1.13)$$

$$\frac{d}{d\alpha_1}(h_2\chi_{22}) - \frac{h_2}{r_1}\frac{d\varepsilon_{22}}{d\alpha_1} - \chi_{11}\frac{dh_2}{d\alpha_1} + (\varepsilon_{11} - \varepsilon_{22})\frac{1}{r_1}\frac{dh_2}{d\alpha_1} + 2\varepsilon_{13}\frac{h_1h_2}{r_1r_2} - \frac{h_2}{r_2}\frac{d\varepsilon_{33}}{d\alpha_1} +$$
$$\frac{2}{r_1}\frac{dh_2}{d\alpha_1}[(\varepsilon_{11} - \varepsilon_{22})(r_1\chi_{11} - \varepsilon_{11}) - 2\varepsilon_{13}^2] + 2\varepsilon_{13}(r_1\chi_{11} + r_2\chi_{22} -$$
$$\varepsilon_{11} + \varepsilon_{22})\frac{h_1h_2}{r_1r_2} - \frac{h_2}{r_2}\frac{d\varepsilon_{22}}{d\alpha_1}\left[\frac{r_2}{r_1}(r_1\chi_{11} - \varepsilon_{11}) + r_2\chi_{22} - \varepsilon_{22}\right] = 0$$

which are the finite strain compatibility equations of the middle surface of a shell of revolution. After substituting (1.11) and (1.12) therein, the compatibility equations can be reduced to the form

$$(1 + e_{22})\frac{d\Psi}{d\alpha_1} - \left(\frac{de_{11}}{d\alpha_1} + 2e_{13}\frac{h_1}{r_1}\right)\Psi = 0 \qquad (1.14)$$
$$\left(1 + e_{22} - e_{11} + \frac{r_1}{h_1}\frac{d\vartheta}{d\alpha_1}\right)\Psi = 0$$
$$\left(\Psi = \frac{1}{h_1}\frac{d}{d\alpha_1}(h_2e_{22}) - \frac{e_{11}}{h_1}\frac{dh_2}{d\alpha_1} + (\vartheta - 2e_{13})\frac{h_2}{r_2}\right)$$

This latter expression corresponds to the linear part of the small strain compatibility equation for axisymmetric shells of revolution [3].

Taking into account that for arbitrary strains the factor in the parenthesis is not zero, there follows from the second equation in (1.14)

$$\Psi = 0 \qquad (1.15)$$

The first equation in (1.14) evidently becomes an identity upon conservation of the condition (1.15).

Therefore, the finite strain compatibility equations of axisymmetrically deformed shells of revolution will be satisfied identically if the linear compatibility equation (1.15) is satisfied.

**2. Determination of the geometric characteristics of a deformed shell.** Let us use the dependence between the Lamé coefficients in the deformed and undeformed states [4]

$$H_1 = h_1\sqrt{1 + 2\varepsilon_{11}}, \quad H_2 = h_2\sqrt{1 + 2\varepsilon_{22}} \qquad (2.1)$$

Substituting (1.11) into (2.1), expanding the factors for $h_1$ and $h_2$ in power series and limiting ourselves to squares of the small strain components, we obtain

$$H_1 = h_1[1 + e_{11} + {}^1/_2(\vartheta - 2e_{13})^2], \quad H_2 = h_2(1 + e_{22}) \qquad (2.2)$$

To determine the radii of curvature of the deformed surface $R_1$ and $R_2$, we use the formulas presented in the monograph [5]

$$\frac{1}{R_1} = -\frac{1}{H_1}\frac{\partial \mathbf{E}_1}{\partial \alpha_1}\cdot\mathbf{E}_3, \quad \frac{1}{R_2} = -\frac{1}{H_2}\frac{\partial \mathbf{E}_2}{\partial \alpha_1}\cdot\mathbf{E}_3 \qquad (2.3)$$

Here $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are basis directions of the state of strain.

Let us connect the radius vector of the strain state $\mathbf{R}$ to the radius-vector of the initial state $\mathbf{r}$ and the displacement vector $\mathbf{u}$

$$\mathbf{R} = \mathbf{r} + \mathbf{u} \tag{2.4}$$

This equality permits expression of the unit basis vectors of the state of strain in terms of the basis vectors of the undeformed state $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

Following [1], we write ($\mathbf{R}_1$, $\mathbf{R}_2$ are covariant basis vectors)

$$\mathbf{E}_1 = \frac{\mathbf{R}_1}{H_1}, \quad \mathbf{E}_2 = \frac{\mathbf{R}_2}{H_2} \quad \left(\mathbf{R}_1 = \frac{\partial \mathbf{R}}{\partial \alpha_1}, \ \mathbf{R}_2 = \frac{\partial \mathbf{R}}{\partial \alpha_2}\right) \tag{2.5}$$

Expanding the displacement vector $\mathbf{u}$ with respect to basis vectors of the initial state and taking the derivative of $\mathbf{R}$ with respect to the coordinate $\alpha_1$, we obtain

$$\mathbf{R}_1 = \frac{\partial \mathbf{r}}{\partial \alpha_1} + \frac{\partial u_1}{\partial \alpha_1}\mathbf{e}_1 + \frac{\partial u_2}{\partial \alpha_1}\mathbf{e}_2 + \frac{\partial u_3}{\partial \alpha_1}\mathbf{e}_3 + u_1\frac{\partial \mathbf{e}_1}{\partial \alpha_1} + u_2\frac{\partial \mathbf{e}_2}{\partial \alpha_1} + u_3\frac{\partial \mathbf{e}_3}{\partial \alpha_1} \tag{2.6}$$

$$(\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)$$

Furthermore, using the derivation formulas for an axisymmetric surface [5]

$$\frac{\partial \mathbf{e}_1}{\partial \alpha_1} = -\frac{h_1}{r_1}\mathbf{e}_3, \quad \frac{\partial \mathbf{e}_2}{\partial \alpha_1} = 0, \quad \frac{\partial \mathbf{e}_3}{\partial \alpha_1} = \frac{h_1}{r_1}\mathbf{e}_1$$

we convert (2.6) to

$$\mathbf{R}_1 = h_1[(1 + e_{11})e_1 - (\vartheta - 2e_{13})\mathbf{e}_3] \tag{2.7}$$

It is possible to obtain analogously

$$\mathbf{R}_2 = h_2(1 + e_{22})\mathbf{e}_2 \tag{2.8}$$

In these latter formulas

$$e_{11} = \frac{1}{h_1}\frac{du_1}{d\alpha_1} + \frac{u_3}{r_1}, \quad e_{22} = \frac{u_1}{h_1 h_2}\frac{dh_2}{d\alpha_1} + \frac{u_3}{r_2}$$

$$\vartheta - 2e_{13} = \frac{u_1}{r_1} - \frac{1}{h_1}\frac{du_3}{d\alpha_1}$$

To determine the unit basis vectors of the state of strain, it is sufficient to multiply (2.7) and (2.8), respectively, by $\sqrt{1 + 2\varepsilon_{11}}$ and $\sqrt{1 + 2\varepsilon_{22}}$ in conformity with (2.1) and (2.5). Expanding the last factors in power series, and multiplying, we obtain to the accuracy used

$$\mathbf{E}_1 = [1 - {}^1/_2(\vartheta - 2e_{13})^2]\mathbf{e}_1 - (1 - e_{11})(\vartheta - 2e_{13})\mathbf{e}_3, \quad \mathbf{E}_2 = \mathbf{e}_2 \tag{2.9}$$

We obtain the third basis vector $\mathbf{E}_3$ by multiplying $\mathbf{E}_1$ vectorally by $\mathbf{E}_2$, which yields

$$\mathbf{E}_3 = (1 - e_{11})(\vartheta - 2e_{13})\mathbf{e}_1 + [1 - {}^1/_2(\vartheta - 2e_{13})^2]\mathbf{e}_3 \tag{2.10}$$

Now substituting (2.4), (2.5), (2.9), (2.10) into (2.3), differentiating and limiting ourselves to the squares of the deformation, we obtain

$$\frac{1}{R_1} = \frac{1}{r_1}\left[1 - e_{11} + e_{11}^2 - \frac{1}{2}(\vartheta - 2e_{13})^2\right] + \frac{1}{h_1}\left[(1 - 2e_{11}) \times \frac{d}{d\alpha_1}(\vartheta - 2e_{13}) - (\vartheta - 2e_{13})\frac{de_{11}}{d\alpha_1}\right] \tag{2.11}$$

$$\frac{1}{R_2} = \frac{1}{r_2}\left[1 - e_{22} + e_{22}^2 - \frac{1}{2}(\vartheta - 2e_{13})^2\right] + \frac{1}{h_1 h_2}\frac{dh_2}{d\alpha_1}(\vartheta - 2e_{13}) \times (1 - e_{11} - e_{22})$$

If there is no shear strain $(e_{13} = 0)$, then

$$\frac{1}{R_1} = \frac{1}{r_1}\left(1 - e_{11} + e_{11}^2 - \frac{1}{2}\vartheta^2\right) + \varkappa_{11}(1 - 2e_{11}) - \frac{\vartheta}{h_1}\frac{de_{11}}{d\alpha_1}$$

$$\frac{1}{R_2} = \frac{1}{r_2}\left(1 - e_{22} + e_{22}^2 - \frac{1}{2}\vartheta^2\right) + \varkappa_{22}(1 - e_{11} - e_{22})$$

In order for the middle surface of the shell obtained after deformation to remain continuous, it is necessary that the Lamé coefficients and its radii of curvature satisfy the Codazzi — Gauss conditions (1. 9), which are written for the state of strain in the form

$$\frac{d}{d\alpha_1}\left(\frac{H_2}{R_2}\right) - \frac{1}{R_1}\frac{dH_2}{d\alpha_1} = 0, \quad \frac{d}{d\alpha_1}\left(\frac{1}{H_1}\frac{dH_2}{d\alpha_1}\right) + \frac{H_1 H_2}{R_1 R_2} = 0 \qquad (2.12)$$

Substituting the expressions (2. 2) for the Lamé coefficients and (2. 11) for the radii of curvature for the state of strain into the last relationships, and subtracting expressions corresponding to the Codazzi — Gauss conditions for the state without strain (1. 9) from (2. 12), we arrive at two equations after tedious manipulations.

$$\frac{d}{d\alpha_1}[(1 - e_{11})\Psi] = 0, \quad \left[1 - e_{11} + \frac{r_1}{h_1}\frac{h}{d\alpha_1}(\vartheta - 2e_{13})\right]\Psi = 0 \qquad (2.13)$$

where the quantity $\Psi$ is determined by the equality in the parentheses in (1. 14).

The nonlinear equations (2. 13) will evidently be satisfied if the linear equation (1. 15) is satisfied. Therefore, the expressions obtained to determine the Lamé coefficients (2. 2) and the radii of curvature (2. 11) for the strained shell correspond to the continuity conditions for its middle surface.

Let the coordinate $\alpha_1$ of some point $a$ (Fig. 1) corresponds, in the undeformed state, to an angle $\omega$ measured from the axis of symmetry to the direction of the normal $\mathbf{e}_3$ to the shell middle surface. An angle $\Omega$ which we define as follows

$$\Omega = \omega + \Delta\omega, \ \Delta\omega = \arccos\,(\mathbf{E}_3 \cdot \mathbf{e}_3)$$

corresponds to this same point (the point A ) with the coordinate $\alpha_1$ in the state of strain.

Using (2. 10), we obtain

$$\Delta\omega = \arccos\,[1 - {}^1/_2\,(\vartheta - 2e_{13})^2]$$

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